

# 6.1 Area Between Curves

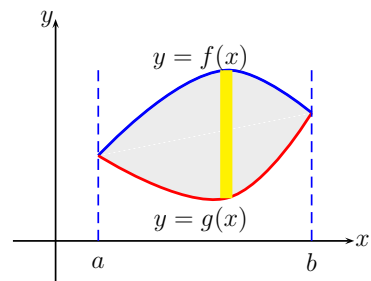
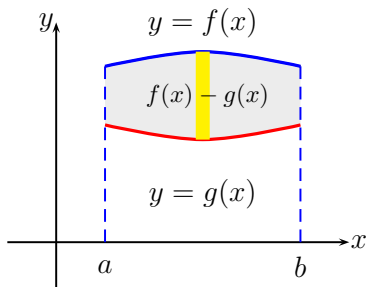
February 21, 2015

## 1 Area Between Curves

### 1.1 Vertical Rectangles Method

*Definition 1: [Area of a Region Between Two Curves]*

If  $f$  and  $g$  are continuous functions on the interval  $[a, b]$  and if  $f(x) \geq g(x)$  for all  $x$  in  $[a, b]$ , then the area of the region bounded above by  $y = f(x)$ , bounded below by  $y = g(x)$ , On the left by  $x = a$ , and on the right by  $x = b$  is  $A = \int_a^b [f(x) - g(x)] dx$ . We usually use the notation  $A = \int_a^b [y_{\text{top}} - y_{\text{bottom}}] dx$ .



### Guidelines for Finding the Area-Vertical Rectangles Method

1. Sketch the region
2. Sketch a typical vertical rectangle (line) through the region at an arbitrary point  $x$  connecting the top and bottom boundaries.
3. The  $y$ -coordinate of the top endpoint of the line sketched in (2) is  $y_{\text{top}}$  and the bottom is  $y_{\text{bottom}}$ .
4. To Find the limits of the integration, move the line you sketched in (2) left to right. The leftmost position on the  $x$ -axis at which the line intersects the region is the lower limit and the rightmost is the upper limit.



5. Compute  $A = \int_{\text{lower}}^{\text{upper}} [y_{\text{top}} - y_{\text{bottom}}] dx$ .

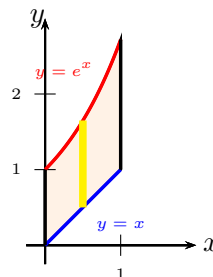
**Example 1:** Find the area of the region bounded above by  $y = x^2 + 3$ , bounded below by  $y = x$ , and bounded on the sides by the lines  $x = 1$  and  $x = -1$ .

**Solution:**

Here the top curve is  $y = x^2 + 3$  and the bottom curve is  $y = x$ .

Also we have  $a = -1$  and  $b = 1$ . Hence

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)] dx \\ &= \int_{-1}^1 [x^2 + 3 - x] dx \\ &= \left[ \frac{x^3}{3} + 3x - \frac{x^2}{2} \right]_{-1}^1 \\ &= \left[ \frac{1}{3} + 3 - \frac{1}{2} \right] - \left[ -\frac{1}{3} - 3 - \frac{1}{2} \right] \\ &= \frac{1}{3} + 3 - \frac{1}{2} + \frac{1}{3} + 3 + \frac{1}{2} \\ &= \frac{2}{3} + 6 + \frac{1}{2} = \frac{14}{3} + \frac{1}{2} = \frac{28}{6} + \frac{3}{6} = \frac{31}{6} \end{aligned}$$

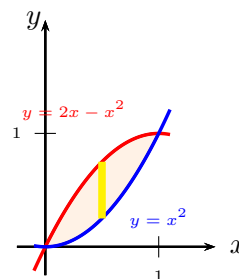


**Example 2:** Find the area of the region enclosed by the parabolas  $y = x^2$  and  $y = 2x - x^2$ .

**Solution:**

Here the top curve is  $y = 2x - x^2$  and the bottom curve is  $y = x^2$ . The limits  $a$  and  $b$  will be the  $x$ -coordinates of the intersections of the two curves. To find the limits we set

$$\begin{aligned} 2x - x^2 &= x^2 \\ 2x - 2x^2 &= 0 \\ 2x(1 - x) &= 0 \\ x &= 0, 1 \end{aligned}$$



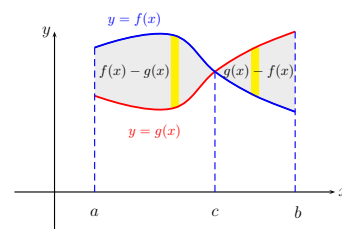
Thus we have  $a = 0$  and  $b = 1$ . Hence

$$\begin{aligned} A &= \int_0^1 [2x - x^2 - x^2] dx \\ &= 2 \int_0^1 [x - x^2] dx \\ &= 2 \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 \left[ \frac{1}{2} - \frac{1}{3} \right] = \frac{2}{3} \end{aligned}$$



**Definition 2: [Area of a Region Between Two Intersecting Curves at More than Two Points]**

If  $f$  and  $g$  are continuous functions on the interval  $[a, b]$ . Let  $a \leq c \leq b$ . Suppose that  $f(x) \geq g(x)$  for all  $x$  in  $[a, c]$ , and  $g(x) \geq f(x)$  for all  $x$  in  $[c, b]$  (Figure 4.1.5). Then the **Area** of the region bounded by  $y = f(x)$ ,  $y = g(x)$ , and the lines  $x = a$ , and  $x = b$  is

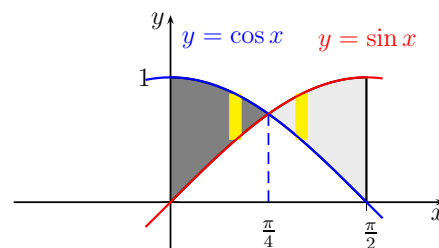


$$A = \int_a^c [f(x) - g(x)] dx + \int_c^b [g(x) - f(x)] dx.$$

**Example 3:** Find the area of the region bounded by the curves  $y = \sin x$ ,  $y = \cos x$ ,  $x = 0$ , and  $x = \frac{\pi}{2}$ .

**Solution:**

The two curves intersect when  $\sin x = \cos x$ , that is, when  $x = \frac{\pi}{4}$ . Now, since  $\cos x \geq \sin x$  when  $0 \leq x \leq \frac{\pi}{4}$  and  $\sin x \geq \cos x$  when  $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$ . Then the area is



$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} [\cos x - \sin x] dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\sin x - \cos x] dx \\ &= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\sin x - \cos x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \left[ \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (1 - 0) \right] + \left[ (-0 - 1) - \left( -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \right] \\ &= \left[ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] \\ &= \frac{4}{\sqrt{2}} - 2 = 2\sqrt{2} - 2. \end{aligned}$$

**Example 4:** Find the area of the region enclosed by  $y^2 = x$  and  $x - 2y = 3$ .

**Solution:**

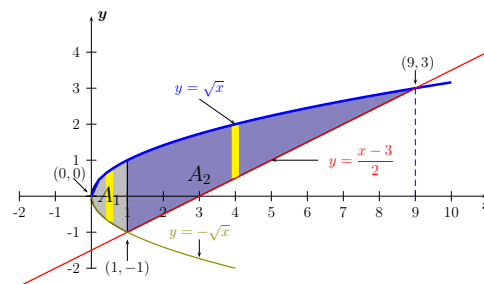
We start with solving the equations  $y^2 = x$  and  $x - 2y = 3$ .

$$y^2 - 2y = 3$$

$$y^2 - 2y - 3 = 0$$

$$(y - 3)(y + 1) = 0 \Rightarrow y = 3, -1. \text{ Thus } x = 9, 1.$$

Hence the points of intersection are  $(1, -1)$  and  $(9, 3)$ . Knowing the points of intersection does not give us the right limits for the integral. Observe that  $x$  moves from 0 to 9. When  $0 \leq x \leq 1$ , the top





curve is  $y = \sqrt{x}$  and the bottom curve is  $y = -\sqrt{x}$ , and when  $1 \leq x \leq 9$ , the top curve is  $y = \sqrt{x}$  and the bottom curve is  $y = \frac{x-3}{2}$ . Hence  $A = A_1 + A_2$ .

$$\begin{aligned} A_1 &= \int_0^1 [\sqrt{x} - (-\sqrt{x})] dx = \int_0^1 [2x^{\frac{1}{2}}] dx \\ &= \left[ \frac{4}{3}x^{\frac{3}{2}} \right]_0^1 = \left[ \frac{4}{3} \right] \\ A_2 &= \int_1^9 \left[ \sqrt{x} - \left( \frac{x-3}{2} \right) \right] dx = \int_1^9 \left[ x^{\frac{1}{2}} - \frac{1}{2}x + \frac{3}{2} \right] dx \\ &= \left[ \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^2 + \frac{3}{2}x \right]_1^9 \\ &= \left[ \left( \frac{54}{3} - \frac{81}{4} + \frac{27}{2} \right) - \left( \frac{2}{3} - \frac{1}{4} + \frac{3}{2} \right) \right] \\ &= \frac{28}{3}. \end{aligned}$$

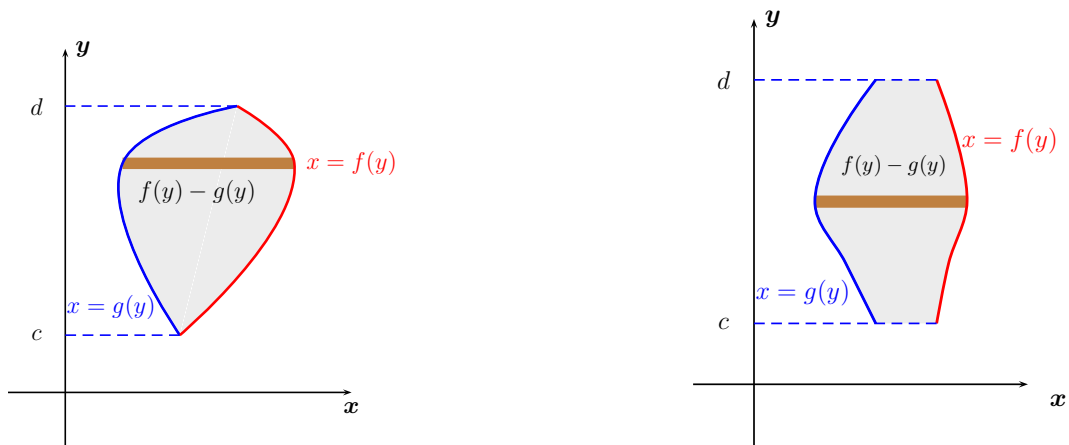
Now,  $A = \frac{28}{3} + \frac{4}{3} = \frac{32}{3}$ .

## 1.2 Horizontal Rectangles Method

### *Definition 3: [Area of a Region Between Two Curves]*

If  $f$  and  $g$  are continuous functions on the interval  $[c, d]$ , and if  $f(y) \geq g(y)$  for all  $y$  in  $[c, d]$ , then the **Area** of the region bounded on the right by  $x = f(y)$ , on the left by  $x = g(y)$ , on the top by the line  $y = d$ , and on the bottom by  $y = c$  is  $A = \int_c^d [f(y) - g(y)] dy$ .

We usually use the notation  $A = \int_{\text{lower}}^{\text{upper}} [x_{\text{Right}} - x_{\text{Left}}] dy$ .





### 1.2.1 Guidelines for Finding the Area-Horizontal Rectangles Method

1. Sketch the region
2. Sketch a typical horizontal rectangle (line) through the region at an arbitrary point  $y$  connecting the right and left boundaries.
3. The  $x$ -coordinate of the right endpoint of the line sketched in (2) is  $x_{\text{Right}}$  and the left is  $x_{\text{Left}}$ .
4. To Find the limits of the integration, move the line you sketched in (2) bottom to top. The bottommost position on the  $y$ -axis at which the line intersects the region is the lower limit and the topmost is the upper limit.

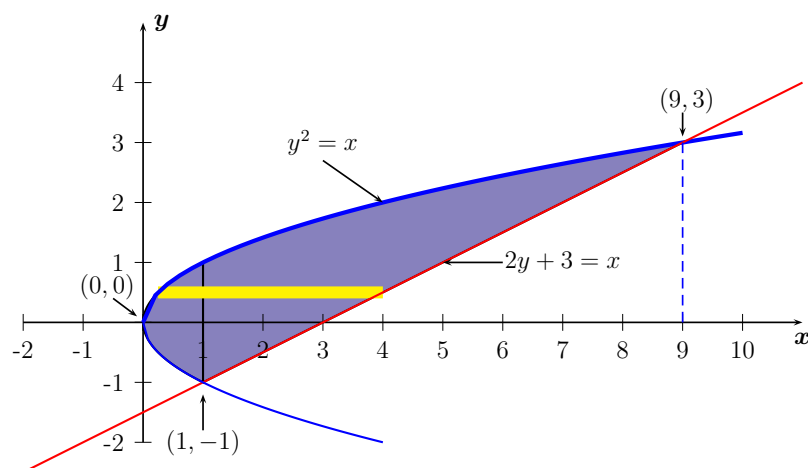
5. Compute  $A = \int_{\text{lower}}^{\text{upper}} [x_{\text{Right}} - x_{\text{bottom}}] dy$ .

**Example 5:** Find the area of the region enclosed by  $y^2 = x$  and  $x - 2y = 3$ .

**Solution:**

We start with solving the equations  $y^2 = x$  and  $x - 2y = 3$ .

$$\begin{aligned}y^2 - 2y &= 3 \\y^2 - 2y - 3 &= 0 \\(y - 3)(y + 1) &= 0 \Rightarrow y = 3, -1. \text{ Thus } x = 9, 1.\end{aligned}$$



Now, it is clear that  $y$  moves from  $-1$  to  $3$  and we have  $x = 2y + 3$  as the right curve and  $x = y^2$  as



the left curve. Hence

$$\begin{aligned} A &= \int_c^d [f(y) - g(y)] dy \\ &= \int_{-1}^3 [2y + 3 - y^2] dy \\ &= \left[ y^2 + 3y - \frac{1}{3}y^3 \right]_{-1}^3 \\ &= \left[ (9 + 9 - 9) - \left( 1 - 3 + \frac{1}{3} \right) \right] \\ &= \left[ 9 - 1 + 3 - \frac{1}{3} \right] \\ &= \left[ 11 - \frac{1}{3} \right] = \frac{32}{3}. \end{aligned}$$

## 6.2 Volume

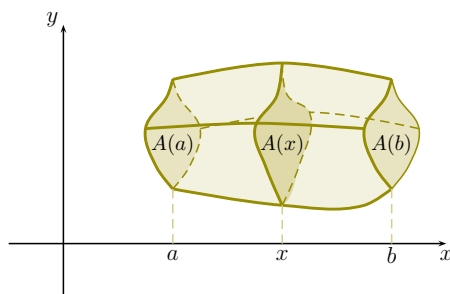
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### 2 Volume: Cross Sections, Disks and Washers

#### 2.1 Volumes by Cross Sections

*Definition 1: [Volume]*

Let  $S$  be a solid bounded by two parallel planes perpendicular to the  $x$ -axis at  $x = a$ , and  $x = b$ . If, for each  $x$  in  $[a, b]$ , the cross-sectional (the intersection of  $S$  with a plane at  $x$ - slicing  $S$  at  $x$ -) area of  $S$  perpendicular to the  $x$ -axis is given by  $A(x)$ , where  $A(x)$  is continuous on  $[a, b]$ , then the volume of  $S$  is  $V = \int_a^b A(x) dx$ .



A similar definition can be given for a solid bounded by two parallel planes perpendicular to the  $y$ -axis.

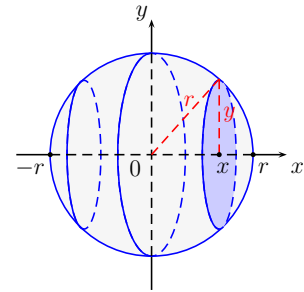
**Example 1:** Prove that the volume of a sphere of radius  $r$  is  $V = \frac{4}{3}\pi r^3$ .

**Solution:**

If we place the sphere so that its center is at the origin then at each  $x$  the cross section by planes perpendicular to the  $x$ -axis is a circle whose radius (from Pythagorean theorem) is  $y = \sqrt{r^2 - x^2}$ . Hence the cross-sectional area is  $A(x) = \pi y^2 = \pi(r^2 - x^2)$ , where  $x$  moves from  $-r$  to  $r$ . Therefore



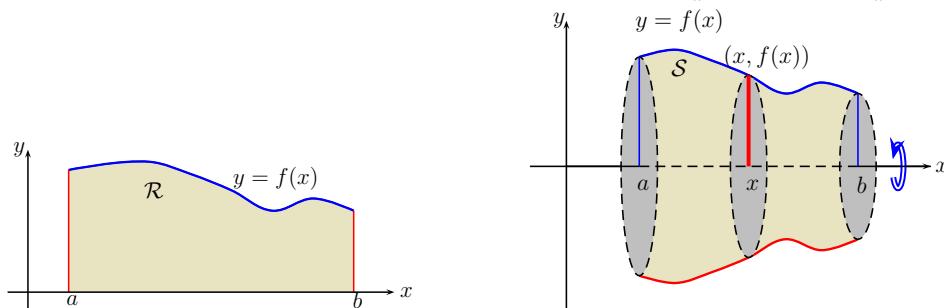
$$\begin{aligned}
 V &= \int_{-r}^r A(x) dx \\
 &= \int_{-r}^r \pi(r^2 - x^2) dx \\
 &= \pi \left[ r^2x - \frac{x^3}{3} \right]_{-r}^r \\
 &= \left[ \left( r^3 - \frac{r^3}{3} \right) - \left( -r^3 + \frac{r^3}{3} \right) \right] \\
 &= \frac{4}{3} \pi r^3.
 \end{aligned}$$



## 2.2 The Disk Method

### *Definition 2: [Volume By The Disk Perpendicular To x-Axis ]*

Let  $\mathcal{R}$  be the region bounded by  $y = f(x)$ ,  $y = 0$ ,  $x = a$ , and  $x = b$ . When this region is revolved about  $x$ -axis we get a solid  $S$ . A cross-section through  $x$  perpendicular to the  $x$ -axis is a circular disk with radius equal to  $f(x)$ . Hence the area of the cross-section is  $A(x) = \pi(\text{radius})^2 = \pi [f(x)]^2$ . The volume of the solid  $S$  is given by the following formula  $V = \int_a^b A(x) dx = \int_a^b \pi [f(x)]^2 dx$ .

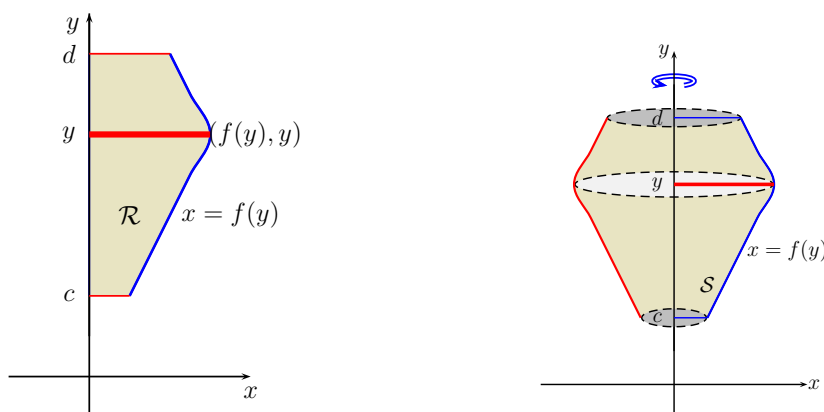


A similar definition is given below when revolving about  $y$ -axis.

### *Definition 3: [Volume By The Disk Perpendicular To y-Axis ]*

Let  $\mathcal{R}$  be the region bounded by  $x = f(y)$ ,  $x = 0$ ,  $y = c$ , and  $y = d$ . When this region is revolved about  $y$ -axis we get a solid  $S$ . A cross-section through  $y$  perpendicular to the  $y$ -axis is a circular disk with radius equal to  $f(y)$ . Hence the area of the cross-section is  $A(y) = \pi(\text{radius})^2 = \pi [f(y)]^2$ . The volume of the solid  $S$  is given by the following formula  $V = \int_c^d A(y) dy = \int_c^d \pi [f(y)]^2 dy$ .





### 2.2.1 Guidelines for Finding the Volume using Disk

We will abbreviate the phrase "integrated with respect to" by "i.w.r."

1. Draw the region and sketch a thin rectangle (line) through the region at  $(x$  or  $y$ ) that is perpendicular to the axis of revolution ( $x$ -axis or  $y$ -axis).
2. If the thin rectangle (line) lies on the  $x$ -axis, then the integration will be with respect to  $x$ . If the thin rectangle (line) lies on the  $y$ -axis, then the integration will be with respect to  $y$ .
3. Label the thin rectangle (line)  $[(x, \text{top}) - (x, \text{bottom})$  if i.w.r.  $x]$  or  $[(\text{left}, y) - (\text{right}, y)$  if i.w.r.  $y]$
4. Find the area of the disk  $[A(x) = \pi(\text{top} - \text{bottom})^2$ -if i.w.r.  $x]$  or  $[A(y) = \pi(\text{right} - \text{left})^2$ -if i.w.r.  $y]$
5. To find the limits move the thin rectangle (line) [left to right if i.w.r.  $x$ , the leftmost is the lower limit and the rightmost is the upper limit] or [bottom to top if i.w.r.  $y$ , the bottommost is the lower limit and the topmost is the upper limit]
6. Compute the integral to get the volume.

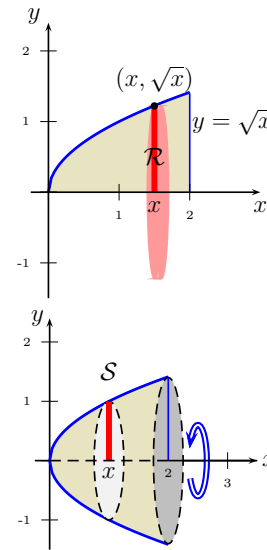
**Example 2:** Find the volume of the solid that is obtained when the region  $y = \sqrt{x}$ ,  $y = 0$ ,  $x = 0$ , and  $x = 2$  is revolved about the  $x$ -axis.

**Solution:**



If we draw a thin rectangle through the region at  $x$  that is perpendicular to the axis of revolution we see it lies on the  $x$ -axis. Hence we should integrate with respect to  $x$ . The cross-section at any  $x \in [0, 2]$  perpendicular to the  $x$ -axis is a disk with radius  $\sqrt{x}$ . Then

$$\begin{aligned} V &= \int_a^b \pi [f(x)]^2 dx \\ &= \int_0^2 \pi [\sqrt{x}]^2 dx \\ &= \int_0^2 \pi [x] dx = \pi \left[ \frac{1}{2} x^2 \right]_0^2 \\ &= \pi \left[ \frac{1}{2} 4 \right] = 2\pi. \end{aligned}$$

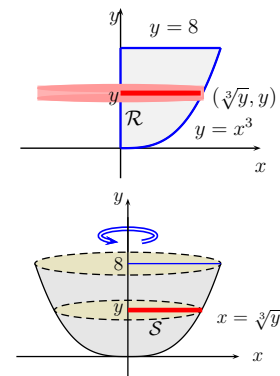


**Example 3:** Find the volume of the solid that is obtained when the region  $y = x^3$ ,  $x = 0$ , and  $y = 8$  is revolved about the  $y$ -axis.

**Solution:**

If we draw a thin rectangle through the region at  $y$  that is perpendicular to the axis of revolution we see it lies on the  $y$ -axis. Hence we should integrate with respect to  $y$ . The cross-section at any  $y \in [0, 8]$  perpendicular to the  $y$ -axis is a disk with radius  $\sqrt[3]{y}$ . Then

$$\begin{aligned} V &= \int_c^d \pi [f(y)]^2 dy = \int_0^8 \pi [\sqrt[3]{y}]^2 dy \\ &= \int_0^8 \pi y^{2/3} dy \\ &= \pi \left[ \frac{5}{3} y^{5/3} \right]_0^8 \\ &= \pi \left[ \frac{5}{3} 8^{5/3} \right] = \frac{160\pi}{3}. \end{aligned}$$

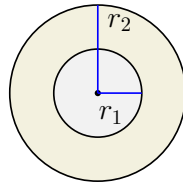


### 2.3 The Washer Method

**Note 1:** Note that the area of an annular [ region bounded by two circles one of radius  $r_1$  and the other of radius  $r_2$  with  $r_1 < r_2$ .] is  $A = \pi[r_2^2 - r_1^2] = \pi[(\text{outer radius})^2 - (\text{inner radius})^2]$ . A definition



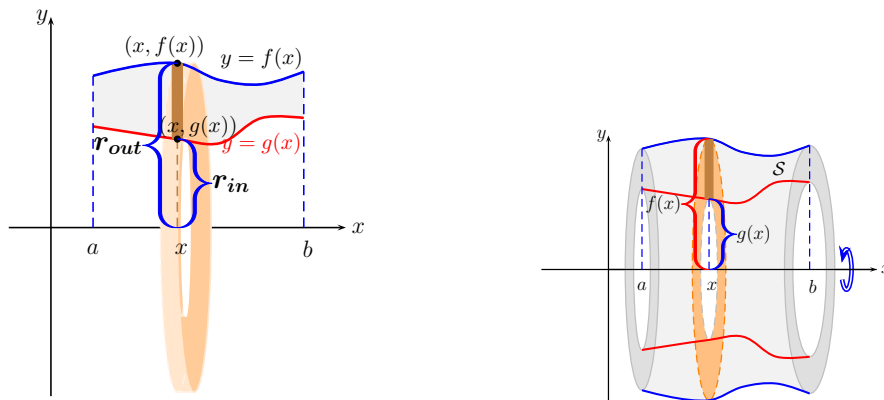
of the volume by the washer perpendicular to  $x$ -axis, a similar definition about the  $y$ -axis should be obtained by the reader.



**Definition 4: [Volume By The Washer Perpendicular To  $x$ -Axis ]**

Let  $\mathcal{R}$  be the region bounded by  $y = f(x)$ ,  $y = g(x)$ ,  $x = a$ , and  $x = b$ , where  $f(x) \geq g(x)$ ,  $\forall x \in [a, b]$ . When this region is revolved about  $x$ -axis we get a solid  $\mathcal{S}$ . A cross-section taken perpendicular to the  $x$ -axis at any point  $x$  is an annular "washer-shaped" with inner radius equal to  $g(x)$  and outer radius equal to  $f(x)$ . Hence the area of the cross-section  $A(x) = \pi[(\text{outer radius})^2 - (\text{inner radius})^2] = \pi[[f(x)]^2 - [g(x)]^2]$ . Then the **Volume** of  $\mathcal{S}$  is given by the following formula

$$V = \int_a^b A(x) dx = \int_a^b \pi[(f(x))^2 - (g(x))^2] dx.$$



**2.3.1 Guidelines for Finding the Volume using Washer**

1. Draw the region and sketch a thin rectangle (line) through the region at  $(x$  or  $y)$  that is perpendicular to the axis of revolution ( $x$ -axis or  $y$ -axis) and extends this line to the axis of revolution.
2. If the thin rectangle (line) lies on the  $x$ -axis, then the integration will be with respect to  $x$ . If the thin rectangle (line) lies on the  $y$ -axis, then the integration will be with respect to  $y$ .
3. Label the thin rectangle (line)  $[(x, \text{top}) - (x, \text{bottom})$  if i.w.r.  $x]$  or  $[(\text{left}, y) - (\text{right}, y)]$  if i.w.r.  $y]$
4. Find the outer radius and the inner radius. The inner radius is the distance from the axis of revolution to the nearest graph to the axis of revolution and the outer radius is the distance from the axis of revolution to the farthest graph. The area of annular  $[A(x)$ -if i.w.r.  $x$  or  $A(y)$ -if



i.w.r.  $y$ ]

$$A = \pi [(outerradius)^2 - (innerradius)^2].$$

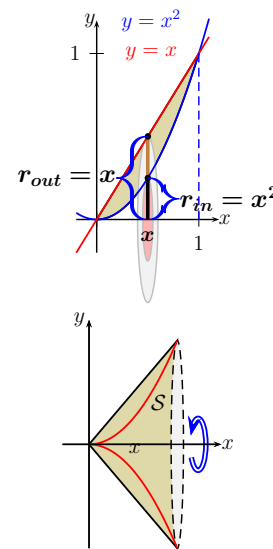
- To find the limits move the thin rectangle (line) [left to right if i.w.r.  $x$ , the leftmost is the lower limit and the rightmost is the upper limit] or [bottom to top if i.w.r.  $y$ , the bottommost is the lower limit and the topmost is the upper limit]
- Compute the integral to get the volume.

**Example 4:** Find the volume of the solid obtained by revolved the region  $y = x$  and  $y = x^2$  about the  $x$ -axis.

**Solution:**

If we draw a thin rectangle through the region at  $x$  that is perpendicular to the axis of revolution we see it lies on the  $x$ -axis. Hence we should integrate with respect to  $x$ . The cross-section at any  $x \in [0, 1]$  perpendicular to the  $x$ -axis is an annular "washer-shaped" with inner radius equal to  $x^2$  and outer radius equal to  $x$ . Then

$$\begin{aligned} V &= \int_0^1 \pi [(x)^2 - (x^2)^2] dx \\ &= \int_0^1 \pi [x^2 - x^4] dx \\ &= \pi \left[ \frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 \\ &= \pi \left[ \frac{1}{3} - \frac{1}{5} \right] = \frac{2\pi}{15}. \end{aligned}$$



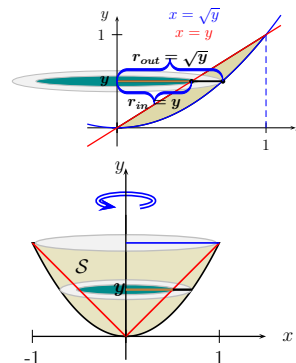
**Example 5:** Find the volume of the solid obtained by revolved the region  $y = x$  and  $y = x^2$  about the  $y$ -axis.



**Solution:**

If we draw a thin rectangle through the region at  $y$  that is perpendicular to the axis of revolution we see it lies on the  $y$ -axis. Hence we should integrate with respect to  $y$ . The cross-section at any  $y \in [0, 1]$  perpendicular to the  $y$ -axis is an annular "washer-shaped" with inner radius equal to  $y$  and outer radius equal to  $\sqrt{y}$ . Then

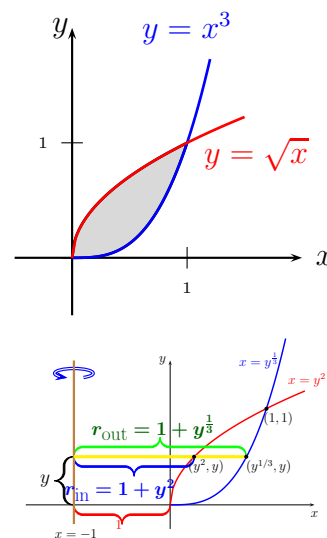
$$\begin{aligned}
 V &= \int_c^d \pi[(\text{outer radius})^2 - (\text{inner radius})^2] dy \\
 &= \int_0^1 \pi[(\sqrt{y})^2 - (y)^2] dy = \int_0^1 \pi [y - y^2] dx \\
 &= \pi \left[ \frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_0^1 \\
 &= \pi \left[ \frac{1}{2} - \frac{1}{3} \right] = \frac{\pi}{6}.
 \end{aligned}$$



**Example 6:** Find the volume of the solid obtained by revolved the region bounded by  $y^2 = x$  and  $y = x^3$  about the  $x = -1$ .

**Solution:**

We have  $y^2 = x \Rightarrow y = \sqrt{x}$  and hence to find the intersection of the curves we set  $x^3 = \sqrt{x}$ . Thus  $x^6 = x$  and hence  $0 = x^6 - x = x(x^5 - 1)$ . Therefore  $x = 0, \Rightarrow y = 0$  or  $x = 1 \Rightarrow y = 1$ . If we draw a thin rectangle through the region at  $y$  that is perpendicular to the axis of revolution  $x = -1$  we see it intersects the  $y$ -axis. Hence we should integrate with respect to  $y$ . The cross-section at any  $y \in [0, 1]$  perpendicular to the  $x = -1$  is an annular "washer-shaped" with inner radius equal to the distance from the  $y$ -axis to the nearest graph  $x = y^2$  plus the distance from the  $y$ -axis to the axis of revolution  $x = -1$ , and hence  $r_{in} = 1 + x_{left} = 1 + y^2$ .

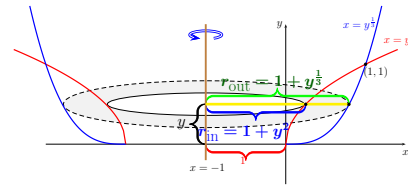




The outer radius equal to the distance from the  $y$ -axis to the farthest graph  $x = y^{1/3}$  plus the distance from the  $y$ -axis to the axis of revolution

$x = -1$ , and hence  $r_{out} = 1 + x_{right} = 1 + y^{1/3}$ . Then

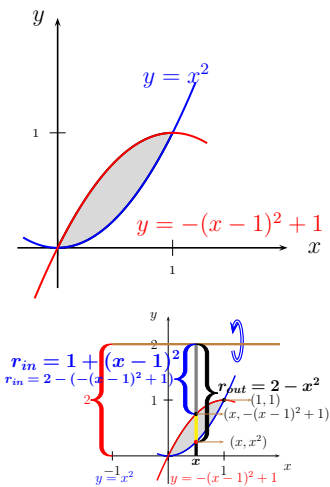
$$\begin{aligned} V &= \int_c^d \pi[(\text{outer radius})^2 - (\text{inner radius})^2] dy \\ &= \int_0^1 \pi[(1 + y^{1/3})^2 - (1 + y^2)^2] dy \\ &= \int_0^1 \pi[2y^{1/3} + y^{2/3} - 2y^2 - y^4] dx \\ &= \pi \left[ \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3} - \frac{2}{3}y^3 - \frac{1}{5}y^5 \right]_0^1 = \frac{37}{30}\pi. \end{aligned}$$



**Example 7:** Find the volume of the solid obtained by revolved the region bounded by  $y = x^2$  and  $y = -(x - 1)^2 + 1$  about the  $y = 2$ .

**Solution:**

To find the intersection of the curves we set  $x^2 = -(x - 1)^2 + 1$ . Thus  $x^2 = -x^2 + 2x$  and hence  $0 = 2x^2 - 2x = 2x(x - 1)$ . Therefore  $x = 0, \Rightarrow y = 0$  or  $x = 1 \Rightarrow y = 1$ . If we draw a thin rectangle through the region at  $x$  that is perpendicular to the axis of revolution  $y = 2$  we see it intersects the  $x$ -axis. Hence we should integrate with respect to  $x$ .

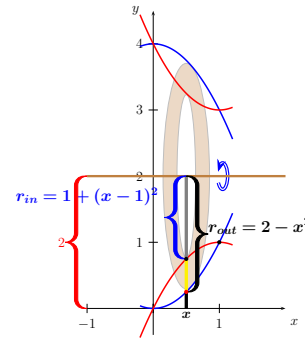




The cross-section at any  $x \in [0, 1]$  perpendicular to the  $y = 2$  is an annular "washer-shaped" with inner radius equal to the distance from the  $x$ -axis to the axis of revolution  $y = 2$ , minus the distance from the  $x$ -axis to the top graph  $y = -(x-1)^2 + 1$ , and hence  $r_{\text{in}} = 2 - (-(x-1)^2 + 1) = 1 + (x-1)^2$ . The outer radius equal to the distance from the  $x$ -axis to the axis of revolution  $y = 2$ , minus the distance from the  $x$ -axis to the bottom graph  $y = x^2$ , and hence  $r_{\text{out}} = 2 - x^2$ .

Then

$$\begin{aligned} V &= \int_c^d \pi[(\text{outer radius})^2 - (\text{inner radius})^2] dy \\ &= \int_0^1 \pi[(2 - x^2)^2 - (1 + (x - 1)^2)^2] dy \\ &= \int_0^1 \pi[-4x^3 + 12x^2 - 8x] dx \\ &= \pi [x^4 - 4x^3 + 4x^2]_0^1 = \pi. \end{aligned}$$



## 6.3 Volume by Cylindrical shells

February 21, 2015

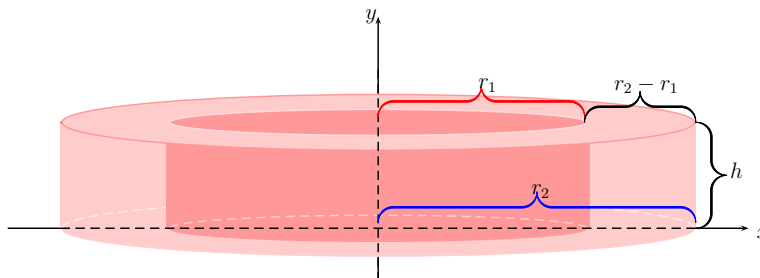
### 3 Volume by Cylindrical shells

#### Cylindrical Shell

*Definition 1: [Cylindrical Shell]*

A **Cylindrical shell** is a solid enclosed by two concentric right-circular cylinders. See Figure 4.3.1 . The volume  $V$  of Cylindrical shell having inner radius  $r_1$  , outer radius  $r_2$  and height  $h$  can be written as

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \\ &= 2\pi r h \Delta r \quad \text{where } r = \frac{r_2 + r_1}{2} \text{ and } \Delta r = r_2 - r_1. \end{aligned}$$

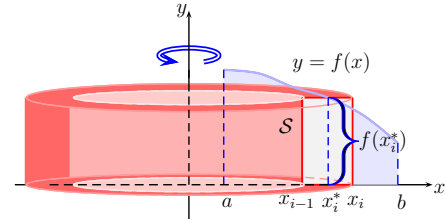
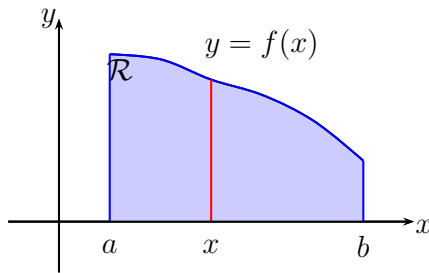


Let  $S$  be the solid obtained by rotating about the  $y$ -axis the region bounded by  $y = f(x)$ ,  $y = 0$ ,  $x = a$ , and  $x = b$ . We partition the interval  $[a, b]$  into  $n$  subinterval  $a = x_0 < x_1 < \dots < x_n = b$  with equal length  $\Delta x$ . Let  $x_i^*$  be the midpoint of  $[x_{i-1}, x_i]$ . If the rectangle with base  $[x_{i-1}, x_i]$  and height  $f(x_i^*)$  is rotated about the  $y$ -axis we get a cylindrical shell with average radius  $x_i^*$ , height  $f(x_i^*)$  and thickness





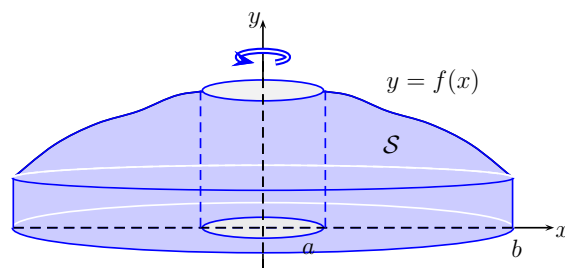
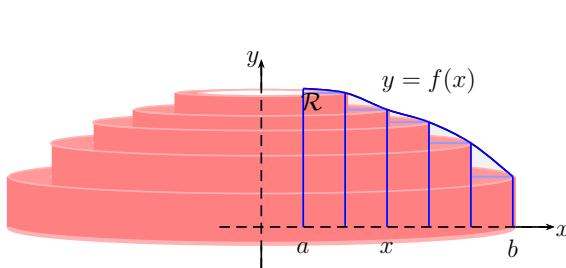
$\Delta x_i = \Delta x$ . Then its volume is  $V_i = 2\pi x_i^* f(x_i^*) \Delta x$  and  $V = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x = \int_a^b 2\pi x f(x) dx$



**Definition 2: [Volume By The Cylindrical Shell Parallel To y-Axis ]**

Let  $S$  be the solid obtained by revolved the region  $y = f(x)$ ,  $y = 0$ ,  $x = a$ ,  $x = b$ . about the  $y$ -axis .Then the volume of is

$$V = \int_a^b 2\pi (\text{shell radius}) (\text{shell height}) dx = \int_a^b 2\pi x f(x) dx$$

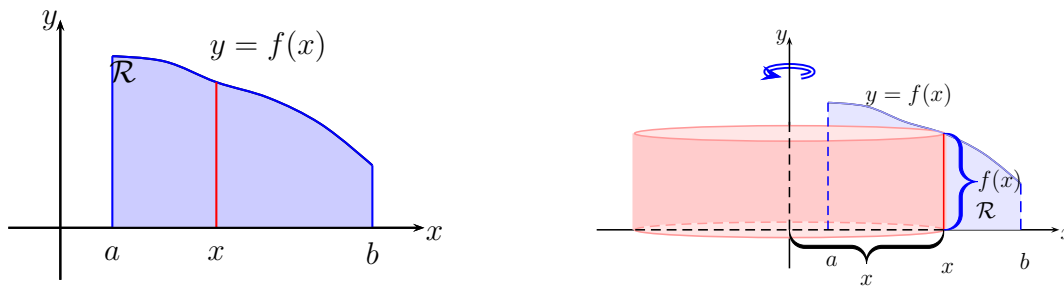


**Definition 3: [Volume By The Cylindrical Shell Parallel To x-Axis ]**

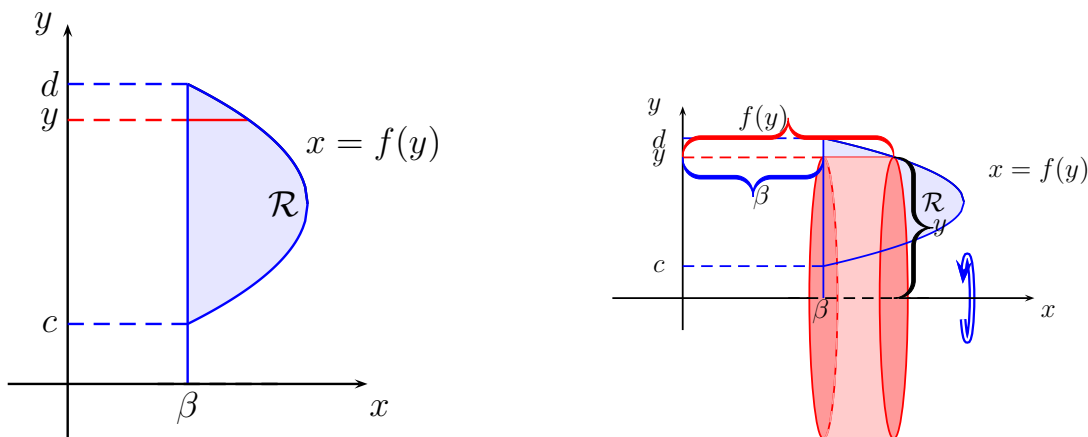
Let  $S$  be the solid obtained by revolved the region  $x = f(y)$ ,  $x = 0$ ,  $y = c$ ,  $y = d$ . about the  $x$ -axis .Then the volume of is

$$V = \int_c^d 2\pi (\text{shell radius}) (\text{shell height}) dy = \int_c^d 2\pi y f(y) dy$$

**Note 1:** Let  $S$  be the solid obtained by rotating about the  $y$ -axis the region bounded by  $y = f(x)$ ,  $y = 0$ ,  $x = a$ , and  $x = b$ . At any  $x$  in the interval  $[a, b]$ , the vertical line segment from the  $x$ -axis to the graph of  $y = f(x)$  can be viewed as a cross-section of the region  $\mathcal{R}$  at  $x$ .(see Figure 4.3.6 and Figure 4.3.7) When the region  $\mathcal{R}$  is revolved about  $y$ -axis, the cross-section at  $x$  generates the surface of a right-circular cylinder of high  $f(x)$  and radius  $x$ . The area of this surface is  $2\pi x f(x)$ . So we can say that the volume by cylindrical shell is the integral of the surface area generated by any arbitrary cross-section of  $\mathcal{R}$  taken parallel to rotation axis  $V = \int_a^b 2\pi x f(x) dx$ .



Similar argument can be made if the axis of revolution is the  $x$ -axis. Let  $S$  be the solid obtained by rotating about the  $x$ -axis the region bounded by  $x = f(y)$ ,  $x = \beta$ ,  $y = c$ , and  $y = d$ . At any  $y$  in the interval  $[c, d]$ , the horizontal line segment from the  $x = \beta$  to the graph of  $x = f(y)$  can be viewed as a cross-section of the region  $\mathcal{R}$  at  $y$ . (see Figure 4.3.8 and Figure 4.3.9) When the region  $\mathcal{R}$  is revolved about  $x$ -axis, the cross-section at  $y$  generates the surface of a right-circular cylinder of height  $f(y) - \beta$  and radius  $y$ . The area of this surface is  $2\pi y[f(y) - \beta]$ . So we can say that the volume by cylindrical shell is the integral of the surface area generated by any arbitrary cross-section of  $\mathcal{R}$  taken parallel to rotation axis  $V = \int_c^d 2\pi y[f(y) - \beta] dy$ .



### Guidelines for Finding the Volume using Cylindrical Shell

We will breviate the phrase "integrated with respect to" by "i.w.r"

1. Draw the region and sketch a thin rectangle (line) through the region at  $(x$  or  $y)$  that is parallel to the axis of revolution.
2. If the thin rectangle (line) lies on the  $x$ -axis, then the integration will be with respect to  $x$ . If the thin rectangle (line) lies on the  $y$ -axis, then the integration will be with respect to  $y$ .
3. Label the thin rectangle (line)  $[(x, \text{top}) - (x, \text{bottom})$  if i.w.r.  $x]$  or  $[(\text{left}, y) - (\text{right}, y)]$  if i.w.r.  $y]$ .
4. Find the radius (the distance from the line to the axis of revolution ) and the height of the cylinder  $[(\text{top} - \text{bottom})$ -if i.w.r.  $x$  or  $(\text{right} - \text{left})$ --if i.w.r.  $y]$ .



5. To find the limits move the thin rectangle (line) [left to right if i.w.r.  $x$ , the leftmost is the lower limit and the rightmost is the upper limit] or [bottom to top if i.w.r.  $y$ , the bottommost is the lower limit and the topmost is the upper limit].

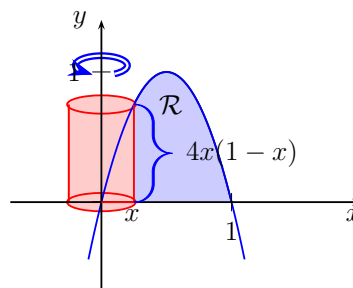
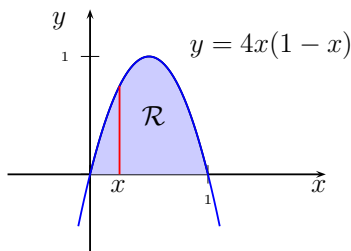
6. Compute the integral  $\int_{\text{lowerlimit}}^{\text{upperlimit}} 2\pi(\text{radius})(\text{height})$  to get the volume.

**Example 1:** find the volume of the solid obtained by rotating the region  $y = 4x(1 - x)$  and  $y = 0$  about the  $y$ -axis

**Solution:**

By solving the equation  $4x(1 - x) = 0$ , we get  $x = 0$ ,  $x = 1$ . If we draw a thin rectangle through the region at  $x$  that is parallel to the axis of revolution ( $y$ -axis) we see it lies on the  $x$ -axis. See Figure 4.3.10. Hence we should integrate with respect to  $x$ . Each vertical line at  $x \in [0, 1]$ , through the region  $\mathcal{R}$  that is parallel to  $y$ -axis generates a cylindrical surface of height  $4x(1 - x)$  and radius  $x$ . See Figure 4.3.11.

Since the area of the surface is  $2\pi x 4x(1 - x)$ , the volume of the solid is

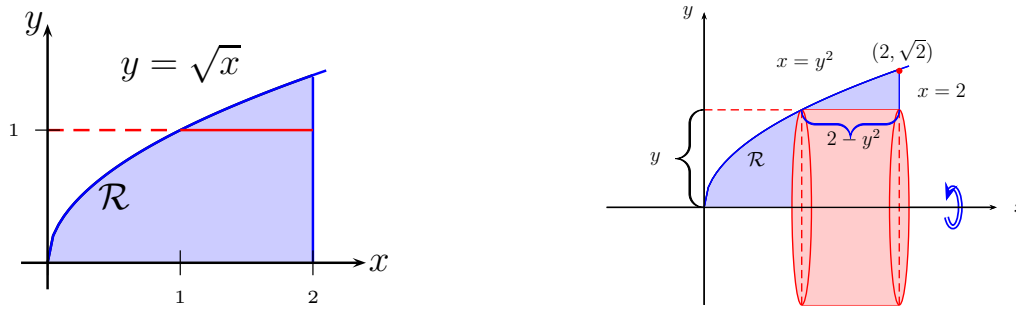
$$V = \int_a^b 2\pi (\text{shell radius}) (\text{shell height}) dx = \int_0^1 2\pi x(4x(1 - x)) dx = 8\pi \int_0^1 x^2 - x^3 dx = 8\pi \left[ \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = \frac{2\pi}{3}.$$


**Example 2:** find the volume of the solid obtained by rotating the region  $y = \sqrt{x}$ ,  $y = 0$ , and  $x = 2$  about the  $x$ -axis.

**Solution:**

Since  $x$  moves from  $x = 0$  to  $x = 2$ , then  $y$  moves from  $y = 0$  to  $y = \sqrt{2}$ . At each  $y \in [0, \sqrt{2}]$ , If we draw a thin rectangle through the region at  $y$  that is parallel to the axis of revolution ( $x$ -axis) we see it lies on the  $y$ -axis. See Figure 4.3.12. Hence we should integrate with respect to  $y$ . Each horizontal line at  $y \in [0, \sqrt{2}]$ , through the region  $\mathcal{R}$  that is parallel to  $x$ -axis generates a cylindrical surface of height  $2 - y^2$  and radius  $y$ . See Figure 4.3.13. Since the area of the surface is  $2\pi y(2 - y^2)$ , the volume of the solid is

$$V = \int_a^b 2\pi (\text{shell radius}) (\text{shell height}) dy = \int_0^{\sqrt{2}} 2\pi y(2 - y^2) dy = 2\pi \left[ 2\frac{1}{2}y^2 - \frac{1}{4}y^4 \right]_0^{\sqrt{2}} = 2\pi.$$



**Example 3:** find the volume of the solid obtained by rotating the region  $y = x^2$  and  $y = x + 2$  about the line  $x = 3$ .

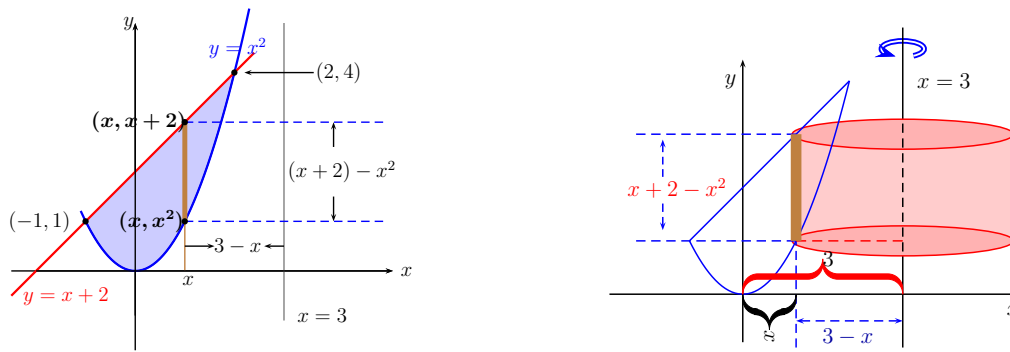
**Solution:**

By solving the equation  $x + 2 = x^2$ ,  $\Rightarrow 0 = x^2 - x - 2 = (x + 1)(x - 2)$  we get  $x = -1$ ,  $x = 2$ . If we draw a thin rectangle through the region at  $x$  that is parallel to the axis of revolution ( $x = 3$ ) we see it lies on the  $x$ -axis. See Figure 4.3.14. Hence we should integrate with respect to  $x$ . Each vertical line at  $x \in [-1, 2]$ , through the region  $\mathcal{R}$  that is parallel to the line  $x = 3$  generates a cylindrical surface of height  $x + 2 - x^2$  and radius  $3 - x$ . See Figure 4.3.15. Now the area of the surface is  $2\pi(3 - x)(x + 2 - x^2)$ , and using the formula

$$V = \int_a^b 2\pi (\text{shell radius}) (\text{shell height}) dx$$

we have the following

$$\begin{aligned} V &= \int_a^b 2\pi (\text{shell radius}) (\text{shell height}) dx \\ &= \int_{-1}^2 2\pi(3 - x)(x + 2 - x^2) dx \\ &= 2\pi \int_{-1}^2 x^3 - 4x^2 + x + 6 dx \\ &= 2\pi \left[ \frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2 + 6x \right]_{-1}^2 \\ &= 2\pi \left[ \left[ \frac{1}{4}2^4 - \frac{4}{3}2^3 + \frac{1}{2}2^2 + 6(2) \right] - \left[ \frac{1}{4}(-1)^4 - \frac{4}{3}(-1)^3 + \frac{1}{2}(-1)^2 + 6(-1) \right] \right] \\ &= \frac{45\pi}{2}. \end{aligned}$$



**Example 4:** find the volume of the solid obtained by rotating the region  $y = \ln x$ ,  $x = 1$ ,  $x = e$  and  $y = 0$  about the line  $y = -1$ .

**Solution:**

Note that  $y = \ln x \Leftrightarrow x = e^y$ . Since  $x$  moves from  $x = 1$  to  $x = e$ , then  $y$  moves from  $y = 0$  to  $y = 1$ . If we draw a thin rectangle through the region at  $y$  that is parallel to the axis of revolution ( $y = -1$ ) we see it lies on the  $y$ -axis. Hence we should integrate with respect to  $y$ . Each horizontal line at  $y \in [0, 1]$ , through the region  $\mathcal{R}$  that is parallel to the line  $y = -1$  generates a cylindrical surface of height  $e - e^y$  and radius  $1 + y$ . See Figure 4.3.17. Now the area of the surface is  $2\pi(1 + y)(e - e^y)$ , hence using the formula  $V = \int_a^b 2\pi (\text{shell radius})(\text{shell height}) dy$  and the fact that  $\int ye^y dy = ye^y - e^y + C$ , we have the following

$$\begin{aligned}
 V &= \int_a^b 2\pi (\text{shell radius})(\text{shell height}) dy \\
 &= \int_0^1 2\pi(1 + y)(e - e^y) dy \\
 &= 2\pi \int_0^1 (e - e^y + ey - ye^y) dy \\
 &= 2\pi \left[ ey - e^y + \frac{e}{2}y^2 - (ye^y - e^y) \right]_0^1 \\
 &= 2\pi \left[ ey + \frac{e}{2}y^2 - ye^y \right]_0^1 \\
 &= \pi e.
 \end{aligned}$$

